QUASI-COMPACTNESS AND UNIFORM ERGODICITY OF POSITIVE OPERATORS

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ABSTRACT

The following converse to the Yosida–Kakutani theorem is proved: If T is a positive operator on a Banach lattice with $\|T^n\|/n \to 0$, then T is quasi-compact if (and only if) the averages of its iterates converge uniformly to a finite-dimensional projection.

Yosida and Kakutani [6] proved that a power-bounded quasi-compact linear operator T is uniformly ergodic, i.e., there exists a (finite-dimensional) projection P such that $\|N^{-1}\Sigma_{j=1}^{N} T^j - P\| \to 0$. Uniform ergodicity, with finite-dimensional fixed points space, does not imply quasi-compactness in general, not even in a Hilbert space [3]. Results obtained for Markov operators [1, 2, 4] suggested the following.

THEOREM 1. *Let T be a positive linear operator on a Banach lattice L, satisfying $\|T^n\|/n \to 0$. Then T is quasi-compact if and only if there is a finite-dimensional projection P such that $\|N^{-1}\Sigma_{j=1}^{N} T^j - P\| \to 0$.*

In case the Banach lattice is complex [5], it is clear that quasi-compactness implies finite-dimensional eigenspaces for each eigenvalue λ of unit modulus.

THEOREM 2. *Let T be a positive linear operator on a complex Banach lattice L with $N^{-1}\Sigma_{j=1}^{N} T^j \to P$ strongly (i.e., T is mean ergodic). If $\dim(PL) = k < \infty$, then $\dim(\{x: Tx = \lambda x\}) \leqq k$ whenever $|\lambda| = 1$.*

PROOF OF THEOREM 2. If $P = 0$, then no eigenvalues λ with $|\lambda| = 1$ are possible. If $Tx = \lambda x$, then $T|x| \geqq |Tx| = |\lambda x| = |x|$, and $N^{-1}\Sigma_{j=1}^{N} T^j|x| \geqq |x|$. Hence

$$\|x\| = \||x|\| \leqq \|N^{-1}\sum_{j=1}^{N} T^j|x|\| \to \|P|x|\| = 0.$$

We now assume $P \neq 0$. If T is irreducible, the result is true by [5, theor. V. 5.2]

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(since the mean ergodic theorem implies existence of positive T^*-invariant functionals, when $P \neq 0$).

We next prove the theorem for the case $P \gg 0$ (i.e., $Py \neq 0$ if $y > 0$). The complex Banach lattice is generated by a real Banach lattice L_0, and we obtain that PL_0 is a sublattice of L_0, since $Ty = y \Rightarrow T|y| \geqq |y|$, and $P(T|y| - |y|) = 0$ implies $T|y| = |y|$. By assumption, PL_0 is finite dimensional, so it is isomorphic (as a Banach lattice) to R^n for some n, by [5, p. 70], and PL is then isomorphic to C^n. Hence $n = k$. If $\{e_1, e_2, \cdots, e_k\}$ is the usual basis of C^k, let $\{y_1, y_2, \cdots, y_k\}$ be the corresponding orthogonal basis of PL. Let J_j be the closed ideal generated by Y_j (i.e., the closure of $\{y \in L : |y| \leqq Ky_j\}$). Each J_j is a sublattice invariant under T, and T_j, the restriction of T to J_j, is irreducible, since T_j is mean ergodic by assumption and PJ_j has one-dimensional range (we use [5, III.8.5]). Thus there is at most a one-dimensional subspace of J_j with eigenvectors corresponding to λ (fixed, with $|\lambda| = 1$). Let $x_j \in J_j$ satisfy $Tx_j = \lambda x_j$, with $x_j \neq 0$ if there is one. $J = \Sigma_{j=1}^k J_j$ is a closed ideal, invariant under T. If $Tx = \lambda x$, then $T|x| \geqq |x|$ and by $P \gg 0$ we have $T|x| = |x|$, showing $|x| \in PL \subset J$. Hence $x \in J$ and $x = \Sigma z_j$ with $z_j \in J_j$. By uniqueness of the decomposition,

$$\sum_{j=1}^{k} \lambda z_j = \lambda x = Tx = \sum_{j=1}^{k} Tz_j \Rightarrow Tz_j = \lambda z_j,$$

so that $z_j = d_j x_j$. Hence $x = \Sigma_{j=1}^k d_j x_j$, and $\dim\{x : Tx = \lambda x\} \leqq k$.

The final step is to reduce the general case to the previous case. Let $R = \{z \in L : P|z| = 0\}$. Setting $R_0 = L_0 \cap R$, we have that $R = R_0 + iR_0$, and R_0 (and R) is invariant under T. The quotient $L_1 = L/R$ is a complex Banach lattice (generated by the real lattice L_0/R_0). The operator S induced on L_1 by T is positive and mean ergodic, with $N^{-1}\Sigma_{j=1}^N S^j \to Q$, where Q is induced on L_1 by P. Clearly $Q \gg 0$, and since $\dim PL = k$, $\dim QL = k$. Suppose $|\lambda| = 1$ and we have $x_1, \cdots, x_{k+1}$ satisfying $Tx_j = \lambda x_j$. Then by applying the previous step to Q, there are $\{\alpha_j\}$ which are not all zero, with $\Sigma_{j=1}^{k+1} \alpha_j \hat{x}_j = 0$, or $x = \Sigma_{j=1}^{k+1} \alpha_j x_j \in R$. But $Tx = \lambda x$, and $T|x| \geqq |x|$ implies $\|x\| \leqq \|P|x|\|$, so $x = 0$. Thus $x_1, \cdots, x_{k+1}$ are not independent, and $\dim\{x : Tx = \lambda x\} \leqq k$.

Proof of Theorem 1. We first assume L to be complex.

Theorem 1 can be obtained by combining Theorem 2 with the result of Lotz and Schaefer [5, p. 331], that T uniformly ergodic as in Theorem 1 has only eigenvalues in $\sigma(T) \cap \{\lambda : |\lambda| = 1\}$, which are all poles. We can then obtain a decomposition of the space which will yield quasi-compactness.

However, instead of studying spectral points as poles, it is possible to look only for closedness of $(\lambda I - T)L$ for each $|\lambda| = 1$, and apply the uniform ergodic theorem of [3] at appropriate places. The embedding technique used in [5] will still be needed. First we prove Theorem 1 for T irreducible, by showing that $(\lambda I - T)L$ is closed for $\lambda \in \sigma(T)$ with $|\lambda| = 1$. The embedding technique is needed when λ is not an eigenvalue of T, in order to obtain (after using suitable quotients) that $(\lambda I - T)L$ is closed. The second step is to prove Theorem 1 when $P \gg 0$ (i.e., $Pf \neq 0$ for $f > 0$), obtaining a decomposition to irreducible components, and showing $(\lambda I - T)L$ closed for $\lambda \in \sigma(T)$, $|\lambda| = 1$. Finally, for general P a suitable quotient yields the same result and we have

$$L = \bigcap_{j=1}^{n} (\lambda_j I - T)L \oplus \sum_{j=1}^{n} \oplus \{x : Tx = \lambda_j x\}.$$

T has spectral radius less than 1 on the first summand, and the last sum is finite-dimensional by Theorem 2.

We now prove the theorem for L real. Let $L_c = L + iL$ be the complexification. T on L_c is also uniformly ergodic, and the previous decomposition, for L_c, yields the decomposition $T = TP_0 + \Sigma_{j=1}^{n} \lambda_j P_{\lambda_j}$, where λ_j are the unimodular eigenvalues of T, $P_{\lambda_j} = \lim N^{-1} \Sigma_{k=1}^{N} \lambda_j^{-k} T^k$, and $P_0 = I - \Sigma_{j=1}^{n} P_{\lambda_j}$ projects on $\bigcap_{j=1}^{n} (\lambda_j I - T)L_c$. But for $|\lambda| = 1$, λ is an eigenvalue if and only if λ^{-1} is an eigenvalue, and $P_{\lambda_j} + P_{\lambda_j^{-1}}$ is a projection leaving the real lattice L invariant, with finite-dimensional range. Thus $\Sigma_{j=1}^{n} P_{\lambda_j}$ is a projection in L with finite-dimensional range (over the reals). Also $P_0 L \subset L$, and the restriction T_0 of T to $P_0 L$ satisfies $\| T_0^n \| \to 0$ (since $r(TP_0) < 1$). Hence T on L is quasi-compact.

COROLLARY. *If T is a positive uniformly ergodic operator satisfying $\| T^n \| / n \to 0$, with $\{x : Tx = x\}$ finite-dimensional, then* $\sup_{n \geq 0} \| T^n \| < \infty$.

This follows from the decomposition obtained in the previous proof.

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